$$\prod_{i=0}^{i<n} [x\cos(\frac{i\pi}{n}) + y\sin(\frac{i\pi}{n})] - z = 0$$

Für n = 12 gilt

$$[x\cos(\frac{0\pi}{12})+y\sin(\frac{0\pi}{12})][x\cos(\frac{1\pi}{12})+y\sin(\frac{1\pi}{12})]$$
$$[x\cos(\frac{2\pi}{12})+y\sin(\frac{2\pi}{12})][x\cos(\frac{3\pi}{12})+y\sin(\frac{3\pi}{12})]$$
$$[x\cos(\frac{4\pi}{12})+y\sin(\frac{4\pi}{12})][x\cos(\frac{5\pi}{12})+y\sin(\frac{5\pi}{12})]$$
$$[x\cos(\frac{6\pi}{12})+y\sin(\frac{6\pi}{12})][x\cos(\frac{7\pi}{12})+y\sin(\frac{7\pi}{12})]$$
$$[x\cos(\frac{8\pi}{12})+y\sin(\frac{8\pi}{12})][x\cos(\frac{9\pi}{12})+y\sin(\frac{9\pi}{12})]$$
$$[x\cos(\frac{10\pi}{12})+y\sin(\frac{10\pi}{12})][x\cos(\frac{11\pi}{12})+y\sin(\frac{11\pi}{12})]-z=0$$

Mit

$$\cos(\frac{0\pi}{12})=\cos(0°)=1$$
$$\cos(\frac{1\pi}{12})=\cos(15°)=\frac{\sqrt{6}+\sqrt{2}}{4}$$
$$\cos(\frac{2\pi}{12})=\cos(30°)=\frac{\sqrt{3}}{2}$$
$$\cos(\frac{3\pi}{12})=\cos(45°)=\frac{\sqrt{2}}{2}$$
$$\cos(\frac{4\pi}{12})=\cos(60°)=\frac{1}{2}$$
$$\cos(\frac{5\pi}{12})=\cos(75°)=\frac{\sqrt{6}-\sqrt{2}}{4}$$
$$\cos(\frac{6\pi}{12})=\cos(90°)=0$$
$$\cos(\frac{7\pi}{12})=\cos(105°)=-\frac{\sqrt{6}-\sqrt{2}}{4}$$
$$\cos(\frac{8\pi}{12})=\cos(120°)=-\frac{1}{2}$$
$$\cos(\frac{9\pi}{12})=\cos(135°)=-\frac{\sqrt{2}}{2}$$
$$\cos(\frac{10\pi}{12})=\cos(150°)=-\frac{\sqrt{3}}{2}$$
$$\cos(\frac{11\pi}{12})=\cos(165°)=-\frac{\sqrt{6}+\sqrt{2}}{4}$$

und

$$\sin(\frac{0\pi}{12})=\sin(0°)=0$$
$$\sin(\frac{1\pi}{12})=\sin(15°)=\frac{\sqrt{6}-\sqrt{2}}{4}$$
$$\sin(\frac{2\pi}{12})=\sin(30°)=\frac{1}{2}$$
$$\sin(\frac{3\pi}{12})=\sin(45°)=\frac{\sqrt{2}}{2}$$
$$\sin(\frac{4\pi}{12})=\sin(60°)=\frac{\sqrt{3}}{2}$$
$$\sin(\frac{5\pi}{12})=\sin(75°)=\frac{\sqrt{6}+\sqrt{2}}{4}$$
$$\sin(\frac{6\pi}{12})=\sin(90°)=1$$
$$\sin(\frac{7\pi}{12})=\sin(105°)=\frac{\sqrt{6}+\sqrt{2}}{4}$$
$$\sin(\frac{8\pi}{12})=\sin(120°)=\frac{\sqrt{3}}{2}$$
$$\sin(\frac{9\pi}{12})=\sin(135°)=\frac{\sqrt{2}}{2}$$
$$\sin(\frac{10\pi}{12})=\sin(150°)=\frac{1}{2}$$
$$\sin(\frac{11\pi}{12})=\sin(165°)=\frac{\sqrt{6}-\sqrt{2}}{4}$$

erhalten wir

$$[1x+0y]\left[\frac{\sqrt{6}+\sqrt{2}}{4}x+\frac{\sqrt{6}-\sqrt{2}}{4}y\right]\left[\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right]\left[\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\right]\left[\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right]$$
$$\left[\frac{\sqrt{6}-\sqrt{2}}{4}x+\frac{\sqrt{6}+\sqrt{2}}{4}y\right][0x+1y]\left[-\frac{\sqrt{6}-\sqrt{2}}{4}x+\frac{\sqrt{6}+\sqrt{2}}{4}y\right]\left[-\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right]$$
$$\left[-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\right]\left[-\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right]\left[-\frac{\sqrt{6}+\sqrt{2}}{4}x+\frac{\sqrt{6}-\sqrt{2}}{4}y\right]-z=0$$

$$x\,y\left[\frac{\sqrt{6}+\sqrt{2}}{4}x+\frac{\sqrt{6}-\sqrt{2}}{4}y\right]\left[\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right]\left[\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\right]\left[\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right]$$
$$\left[\frac{\sqrt{6}-\sqrt{2}}{4}x+\frac{\sqrt{6}+\sqrt{2}}{4}y\right]\left[-\frac{\sqrt{6}-\sqrt{2}}{4}x+\frac{\sqrt{6}+\sqrt{2}}{4}y\right]\left[-\frac{1}{2}x+\frac{\sqrt{3}}{2}y\right]$$
$$\left[-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y\right]\left[-\frac{\sqrt{3}}{2}x+\frac{1}{2}y\right]\left[-\frac{\sqrt{6}+\sqrt{2}}{4}x+\frac{\sqrt{6}-\sqrt{2}}{4}y\right]-z=0$$

Jetzt wird das Ausmultiplizieren etwas komplizierter, deshalb gehen wir schrittweise vor. Wir führen A, B, C, D, E, F, G, H, I und J ein.

$$x\,y[A\cdot B\cdot C\cdot D\cdot E\cdot F\cdot G\cdot H\cdot I\cdot J]-z=0$$

$$A=\frac{\sqrt{6}+\sqrt{2}}{4}x+\frac{\sqrt{6}-\sqrt{2}}{4}y$$
$$B=\frac{\sqrt{3}}{2}x+\frac{1}{2}y$$
$$C=\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y$$
$$D=\frac{1}{2}x+\frac{\sqrt{3}}{2}y$$
$$E=\frac{\sqrt{6}-\sqrt{2}}{4}x+\frac{\sqrt{6}+\sqrt{2}}{4}y$$
$$F=-\frac{\sqrt{6}-\sqrt{2}}{4}x+\frac{\sqrt{6}+\sqrt{2}}{4}y$$
$$G=-\frac{1}{2}x+\frac{\sqrt{3}}{2}y$$
$$H=-\frac{\sqrt{2}}{2}x+\frac{\sqrt{2}}{2}y$$
$$I=-\frac{\sqrt{3}}{2}x+\frac{1}{2}y$$
$$J=-\frac{\sqrt{6}+\sqrt{2}}{4}x+\frac{\sqrt{6}-\sqrt{2}}{4}y$$

Um das Ausmultiplizieren zu vereinfachen stellen wir die Gleichungen etwas um.

$$A=\frac{\sqrt{6}-\sqrt{2}}{4}y+\frac{\sqrt{6}+\sqrt{2}}{4}x$$
$$B=\frac{1}{2}y+\frac{\sqrt{3}}{2}x$$
$$C=\frac{\sqrt{2}}{2}y+\frac{\sqrt{2}}{2}x$$
$$D=\frac{\sqrt{3}}{2}y+\frac{1}{2}x$$
$$E=\frac{\sqrt{6}+\sqrt{2}}{4}y+\frac{\sqrt{6}-\sqrt{2}}{4}x$$
$$F=\frac{\sqrt{6}+\sqrt{2}}{4}y-\frac{\sqrt{6}-\sqrt{2}}{4}x$$
$$G=\frac{\sqrt{3}}{2}y-\frac{1}{2}x$$
$$H=\frac{\sqrt{2}}{2}y-\frac{\sqrt{2}}{2}x$$
$$I=\frac{1}{2}y-\frac{\sqrt{3}}{2}x$$
$$J=\frac{\sqrt{6}-\sqrt{2}}{4}y-\frac{\sqrt{6}+\sqrt{2}}{4}x$$

Jetzt nutzen wir die binomische Formel aus einer Formelsammlung.

$$(a+b)(a-b)=a^2-b^2$$

Danach müssen wir A mit J, B mit I, C mit H, D mit G und E mit F multiplizieren.

$$K=A\cdot J=\left[\frac{\sqrt{6}-\sqrt{2}}{4}y+\frac{\sqrt{6}+\sqrt{2}}{4}x\right]\cdot\left[\frac{\sqrt{6}-\sqrt{2}}{4}y-\frac{\sqrt{6}+\sqrt{2}}{4}x\right]=\frac{(\sqrt{6}-\sqrt{2})^2}{16}y^2-\frac{(\sqrt{6}+\sqrt{2})^2}{16}x^2$$

$$(\sqrt{6}-\sqrt{2})(\sqrt{6}-\sqrt{2})=6-2\sqrt{12}+2=8-2\sqrt{12}$$

$$(\sqrt{6}+\sqrt{2})(\sqrt{6}+\sqrt{2})=6+2\sqrt{12}+2=8+2\sqrt{12}$$

$$K=A\cdot J=\frac{8-2\sqrt{12}}{16}y^2-\frac{8+2\sqrt{12}}{16}x^2$$

$$L=B\cdot I=\left[\frac{1}{2}y+\frac{\sqrt{3}}{2}x\right]\cdot\left[\frac{1}{2}y-\frac{\sqrt{3}}{2}x\right]=\frac{1}{4}y^2-\frac{3}{4}x^2$$

$$M=C\cdot H=\left[\frac{\sqrt{2}}{2}y+\frac{\sqrt{2}}{2}x\right]\cdot\left[\frac{\sqrt{2}}{2}y-\frac{\sqrt{2}}{2}x\right]=\frac{2}{4}y^2-\frac{2}{4}x^2$$

$$N=D\cdot G=\left[\frac{\sqrt{3}}{2}y+\frac{1}{2}x\right]\cdot\left[\frac{\sqrt{3}}{2}y-\frac{1}{2}x\right]=\frac{3}{4}y^2-\frac{1}{4}x^2$$

$$O=E\cdot F=\left[\frac{\sqrt{6}+\sqrt{2}}{4}y+\frac{\sqrt{6}-\sqrt{2}}{4}x\right]\cdot\left[\frac{\sqrt{6}+\sqrt{2}}{4}y-\frac{\sqrt{6}-\sqrt{2}}{4}x\right]=\frac{(\sqrt{6}+\sqrt{2})^2}{16}y^2-\frac{(\sqrt{6}-\sqrt{2})^2}{16}x^2$$

$$O=E\cdot F=\frac{8+2\sqrt{12}}{16}y^2-\frac{8-2\sqrt{12}}{16}x^2$$

Damit kommen wir dem Ergebnis einen Schritt näher.

$$x\,y[K\cdot L\cdot M\cdot N\cdot O]-z=0$$

Im nächsten Schritt multiplizieren wir K mit O.

$$P=K\cdot O=\left[\frac{8-2\sqrt{12}}{16}y^2-\frac{8+2\sqrt{12}}{16}x^2\right]\cdot\left[\frac{8+2\sqrt{12}}{16}y^2-\frac{8-2\sqrt{12}}{16}x^2\right]$$

$$(8-2\sqrt{12})(8+2\sqrt{12})=64-48=16$$

$$(8-2\sqrt{12})(8-2\sqrt{12})=64-32\sqrt{12}+48=112-32\sqrt{12}$$

$$(8+2\sqrt{12})(8+2\sqrt{12})=64+32\sqrt{12}+48=112+32\sqrt{12}$$

$$P=K\cdot O=\frac{16}{256}y^4-\frac{112-32\sqrt{12}}{256}x^2y^2-\frac{112+32\sqrt{12}}{256}x^2y^2+\frac{16}{256}x^4$$

$$P=K\cdot O=\frac{16}{256}y^4-\frac{224}{256}x^2y^2+\frac{16}{256}x^4$$

$$P=K\cdot O=\frac{1}{16}y^4-\frac{14}{16}x^2y^2+\frac{1}{16}x^4$$

Im nächsten Schritt multiplizieren wir L mit M.

$$Q=L\cdot M=\left[\frac{1}{4}y^2-\frac{3}{4}x^2\right]\cdot\left[\frac{2}{4}y^2-\frac{2}{4}x^2\right]$$

$$Q=L\cdot M=\frac{2}{16}y^4-\frac{6}{16}x^2y^2-\frac{2}{16}x^2y^2+\frac{6}{16}x^4$$

$$Q=L\cdot M=\frac{1}{8}y^4-\frac{4}{8}x^2y^2+\frac{3}{8}x^4$$

Im nächsten Schritt multiplizieren wir Q mit N.

$$R=Q\cdot N=\left[\frac{1}{8}y^4-\frac{4}{8}x^2y^2+\frac{3}{8}x^4\right]\cdot\left[\frac{3}{4}y^2-\frac{1}{4}x^2\right]$$

$$R=Q\cdot N=\frac{3}{32}y^6-\frac{1}{32}x^2y^4-\frac{12}{32}x^2y^4+\frac{4}{32}x^4y^2+\frac{9}{32}x^4y^2-\frac{3}{32}x^6$$

$$R=Q\cdot N=\frac{3}{32}y^6-\frac{13}{32}x^2y^4+\frac{13}{32}x^4y^2-\frac{3}{32}x^6$$

Im letzten Schritt müssen wir P mit R multiplizieren.

$$S=P\cdot R=\left[\frac{1}{16}y^4-\frac{14}{16}x^2y^2+\frac{1}{16}x^4\right]\cdot\left[\frac{3}{32}y^6-\frac{13}{32}x^2y^4+\frac{13}{32}x^4y^2-\frac{3}{32}x^6\right]$$

$$S=P\cdot R=\frac{3}{512}y^{10}-\frac{13}{512}x^2y^8+\frac{13}{512}x^4y^6-\frac{3}{512}x^6y^4-\frac{42}{512}x^2y^8+\frac{182}{512}x^4y^6-\frac{182}{512}x^6y^4+\frac{42}{512}x^8y^2$$

$$+\frac{3}{512}x^4y^6-\frac{13}{512}x^6y^4+\frac{13}{512}x^8y^2-\frac{3}{52}x^{10}$$

$$S=P\cdot R=\frac{3}{512}y^{10}-\frac{55}{512}x^2y^8+\frac{198}{512}x^4y^6-\frac{198}{512}x^6y^4+\frac{55}{512}x^8y^2-\frac{3}{512}x^{10}$$

Damit erhält man für n = 12

$$x\,y\left[-\frac{3}{512}x^{10}+\frac{55}{512}x^8y^2-\frac{198}{512}x^6y^4+\frac{198}{512}x^4y^6-\frac{55}{512}x^2y^8+\frac{3}{512}y^{10}\right]-z=0$$

Das Ergebnis der Formel.